Nekrasov. Many faces of characters. SCGP, 2014/4.

Characters $\pi: G \longrightarrow GL(V)$ representation

$\chi_\pi(g) = \mathrm{Trace}_V\, \pi(g)$ (a distribution, in general).

- $\dim V = \infty$, $V = \bigoplus_{k>0} V_k$ weight space decomp.
  $G \ni g = q^{L_0}\tilde{g}$ $\dim V_k < \infty$ $\Delta_k = L_0|_{V_k}$
  $\sum_k \dim V_k\, q^{\Delta_k}$ is convergent for q in some domain.

- Almost all special functions are characters.
  i.e. they solve interesting
  differential equations or difference equations

  eg. $\Delta\, \chi_\pi(g) = c_2(V_\pi)\, \chi_\pi(g)$ or $\underbrace{\mathrm{Tr}_{V\otimes U}\, g}_{\Sigma C^W_{UV}\, \mathrm{Tr}_W g} = (\mathrm{Tr}_V\, g)(\mathrm{Tr}_U\, g)$,

- Where do (V, π) come from?

  Ans: Start with a system with G symmetry,
  and quantize. V: space of states

  $\mathrm{Trace}_V\, \pi(g) = \int \mathcal{D}p\, \mathcal{D}q\, e^{i\int_X p\,dq}$ $x(0)$ $x(1) = g\cdot x(0)$

  Path integral over the space of maps $x: [0,1] \to \mathcal{X}$
  with the boundary condition $g\cdot x(0) = x(1)$.

- If $G \curvearrowright \mathcal{X} \xrightarrow{\mu} \mathfrak{g}^*$ Hamiltonian action,
  $\iota_{V_\xi}\omega = dH_\xi$, $\langle\mu, \xi\rangle =: H_\xi$ $\xi \in \mathrm{Lie}G \rightsquigarrow V_\xi \in \mathrm{Vect}(\mathcal{X})$
  then $\mathrm{Trace}_V\, \pi(e^\xi) = \int \mathcal{D}p\, \mathcal{D}q\, e^{i\int_X p\,dq - i\int_X \langle\mu,\xi\rangle}$

  Path integral over the space of LOOPs $x: S^1 \to \mathcal{X}$

- In some cases, there is a nice math. expression,
  $\mathcal{X}$ Kähler, $[\omega] = c_1(\mathcal{L}) \in H^2(\mathcal{X}, \mathbb{Z})$ $\mathbb{C} \to \mathcal{L} \to \mathcal{X}$

  $\Rightarrow$ $V = H^0(\mathcal{X}, \mathcal{L})$ (assuming vanishing).

  $\mathrm{Trace}_V\, \pi(g) = \int_{\mathcal{X}} \left(e^{c_1(\mathcal{L})}\, Td_{\mathcal{X}}\right)^{G\text{-equiv}} = \sum_{p\in\mathcal{X}^g} (\text{local formula})$ eg. Weyl, or Kac-Weyl formula.

↰ Classical ~ Geometry (sheaf cohomology)

↳ Modern ~ Topology (ordinary cohomology).

There is another way of getting the representations and characters.

$$V = H_{top}(M, \mathbb{C}) \quad \text{deRham cohomology.}$$

It will carry a representation of $G, \mathfrak{g}, Y_\hbar(\mathfrak{g}), U_q(\mathfrak{g})$, even though they do NOT act on M.

$$[\alpha] \in H^\bullet_{dR}(M) \qquad d\alpha = 0 \qquad \Rightarrow \delta_v[\alpha] = 0.$$

$$(\because \ \delta_v \alpha = \mathrm{Lie}_v \alpha = d(\iota_v \alpha) + \iota_v(\cancel{d\alpha}) \in \mathrm{Im}(d))$$

$$g \in G \curvearrowright \mathcal{X} \rightsquigarrow \mathrm{Graph}(g) \subset \mathcal{X} \times \mathcal{X}$$

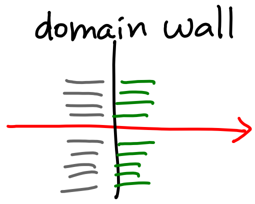
- Hecke correspondence:

$$\begin{array}{ccc} & C \subset M \times M & \\ & \swarrow^{p_1} \quad \searrow^{p_2} & \\ M & & M \end{array}$$

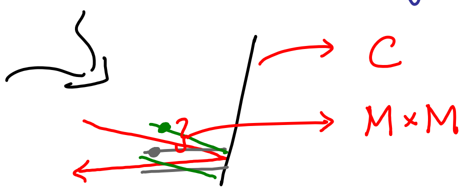
$$\rightsquigarrow H^*(M) \circlearrowleft \qquad \alpha \mapsto p_{2*}(p_1^* \alpha \cap \delta_C)$$

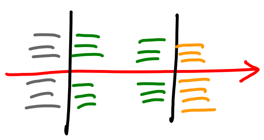
Such C's become generators of some symmetry algebra.

Natural context: 2d σ-model on M.



$\Sigma^2 \longrightarrow M$ $\qquad$ $C \subset M \times M$ ~ defects



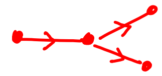


$\rightsquigarrow$ composition $\Rightarrow$ algebra structure.

$$\mathrm{Trace}_V \pi(g) = \sum_\lambda \chi(M^\lambda)\, e^{\langle \lambda, \xi \rangle}, \qquad M^\lambda \overset{\text{subvar.}}{\subset} M$$

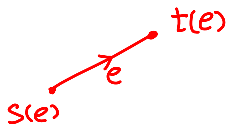
Comparing with geometric case,

$$\mathrm{Trace}_V \pi(g) = \sum_\lambda \chi(\mathcal{X}^\lambda, \mathcal{L})\, e^{\langle \lambda, \xi \rangle}, \quad \mathcal{X}^\lambda \text{ sympl. quotient.}$$

Example : $M = \bigsqcup_V \mathfrak{M}(V,W)$ 

Moduli space of quiver representations

$i \in \mathrm{Vert}_\gamma \mapsto V_i, W_i$ vector spaces.



$$\mathfrak{M}(V,W) = \bigoplus_e T^*\mathrm{Hom}(V_{s(e)}, V_{t(e)}) \oplus \bigoplus_i T^*\mathrm{Hom}(V_i, W_i) /\!\!/^{\text{stable}} \prod_i GL(V_i)$$

for γ ADE Dynkin graph (finite or affine).

$\rightsquigarrow$ 2d Gauge theory w/ $G_{gauge} = \prod_i U(V_i)$

w/ bifundamental + fundamental hypermultiplets.

$\mathcal{N} = (4,4)$, 8 SUSY charge.

- $H^*_{G_W \times \mathbb{C}^\times}(M_W)$ — representation $G_W = \prod_i GL(W_i)$.

of Yangian of $\mathfrak{g}_{ADE}$.

($\exists$ second parameter generalization.)

Eg. $\mathfrak{g} = sl_2$ or A_1, $V = \mathbb{C}^2$

$\pi(g) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \Rightarrow \mathrm{Tr}_V \pi(g) = t + t^{-1}$

For $L\mathfrak{g} \ni g(x)$, $x \in S^1 \Rightarrow \mathrm{Tr}\, g(x) = t(x) + t(x)^{-1}$

Study $\mathcal{N} = 2$ 4d SUSY Gauge theory

$G_{gauge} = U(N)$ $N_f = 2N$ fundamental hypermultiplets

Important field: ϕ adjoint complex Higgs field.

$$Y(x) \triangleq x^N \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} x^{-k} \mathrm{Tr}\, \phi^k\right) \sim \text{“}\det(x-\phi)\text{”}$$

this is NOT a poly. in x.

However, $\langle Y(x) + \mathfrak{q} \cdot p(x) Y(x)^{-1} \rangle$ is poly. in x.

(uses Dyson-Schwinger identities)

where $\mathfrak{q} = \exp\left(-\frac{8\pi^2}{g^2} + i\theta\right)$, $p(x) = \prod_{f=1}^{2N} (x - m_f)$ poly.

$\rightsquigarrow$ poly. maps $\mathbb{C} \longrightarrow G/\mathrm{Ad}G$.

- Quiver $\mathcal{N}=2$ $d=4$ gauge theory.

E.g. $\underset{n_1}{\bullet} \underset{\gamma}{\text{———}} \underset{n_2}{\bullet} \text{———} \underset{n_3}{\bullet}$ $\qquad G = \prod_{i\,\mathrm{Vert}} U(N_i)$

edge$_\gamma$ $\longleftrightarrow$ bifundamental $\qquad\qquad$ fundamental.

$$Y_i(x) = x^N \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} x^{-k} \operatorname{Tr} \phi_i^k\right)$$

$$LG_{ADE} \ni g(x) = \prod_{i \in \mathrm{Vert}} Y_i(x)^{\alpha_i^\vee} \left(q_i\, p_i(x)\right)^{-\lambda_i^\vee}$$

$\alpha_i^\vee$: simple coroot of

$\lambda_i^\vee$: fundamental coweight

$\langle \chi_{V_\omega}(g(x)) \rangle^{4d\ \mathrm{SUSY}}$ is poly. in x

V_ω: finite dim rep. of G_{ADE}.

- 4d $\mathcal{N}=2$ SUSY has Ω-deformation $(\varepsilon_1, \varepsilon_2)$

$\Rightarrow$ above expression also has $(\varepsilon_1, \varepsilon_2)$ deformations.

eg. A_1, $V = \mathbb{C}^2$ case,

$$\langle Y(x+\varepsilon_1+\varepsilon_2) + q\, P(x)\, Y(x)^{-1} \rangle = T_N(x) \qquad U(N)$$

$\rightsquigarrow$ one can derive eqt. obeyed by conformal blocks of 2d CFT.

- $(\varepsilon_1, 0)$-deformation $\Rightarrow$ q-characters of $Y(\mathfrak{g}_{ADE})$.

eg. A_1, $V = \mathbb{C}^2(v_1) \otimes \cdots\cdots \otimes \mathbb{C}^2(v_w)$

$$\chi_V = \sum_{I \cup J = \{1,\dots,w\}} q^{|I|} \prod_{i \in I} \frac{P(x+v_i)}{Y(x+v_i)} \prod_{j \in J} Y(x+v_j+\varepsilon_1+\varepsilon_2) \prod_{\substack{i \in I \\ j \in I}} \frac{(v_i - v_j + \varepsilon_1)(v_i - v_j + \varepsilon_2)}{(v_i - v_j)(v_i - v_j + \varepsilon_1 + \varepsilon_2)}.$$